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SOME PROPERTIES OF THE SHOCK ADIABAT OF QUASITRANSVERSE ELASTIC WAVES *

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It is shown that discontinuities corresponding to points of certain quasitransverse shocks, that are not evolutionary from cuts of the shock adiabat, in an isotropic prestressed elastic medium are a sequence of two evolutionary shocks moving at identical velocity. Two such representations are obtained for certain sections of the shock adiabat. The possibility of representing the non-evolutionary discontinuities in the form of a sequence of evolutionary discontinuities moving at identical velocity in other problems of the mechanics of a continuous medium is discussed.

Quasitransverse shocks are investigated below within the framework of the approximations made in /1,2/, where the set of states (the shock adiabat), into which it is possible to drop from a given initial state by a jump while conserving the conservation laws, was investigated for low-intensity shocks. Segments satisfying the condition of no decrease in entropy and the evolution conditions, i.e., the necessary conditions for correctness of the linearized boundary conditions on the discontinuity /3/, were extracted on the curve representing the shock adiabat. Discontinuities corresponding to shock adiabat segments satisfying the requirement of no entropy decrease but not satisfying the evolutionarity conditions because of an excess in the number of boundary conditions over the number of unknowns in the linearized problem of interation between small perturbations and the discontinuity, are discussed in the present paper.

Representation of the non-evolutionary discontinuities in the form of a sequence of evolutionary discontinuities moving at one velocity can turn out to be useful in solving different selfsimilar problems containing discontinuities.

1. Formulation of the problem. An isotropic elastic medium is given by its internal energy $U(\varepsilon_{ij}, S)$ in the form /1,2/

$$\Phi = \rho_0 U = {}^1/{}_2\lambda I_1^2 + \mu I_2 + \beta I_1 I_2 + \gamma I_3 + \delta I_1^3 + \xi I_2^2 +$$
$$\rho_0 T_0 (S - S_0) + \text{const}$$
$$I_1 = \varepsilon_{ii}, \quad I_2 = \varepsilon_{ij}\varepsilon_{ij}, \quad I_3 = \varepsilon_{ik}\varepsilon_{kj}\varepsilon_{ji}$$
$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial \xi_j} + \frac{\partial w_j}{\partial \xi_i} + \frac{\partial w_k}{\partial \xi_i}\frac{\partial w_k}{\partial \xi_j}\right)$$

Here ε_{ij} are the finite strain tensor components, w_i is the displacement vector, ρ_0 is the density in the unstressed state, S is the entropy, and ξ_i are Lagrange (Cartesian rectangular) coordinates.

In a plane wave with the front $\xi_3 = Wt$ the following displacement gradient components undergo change:

$$\partial w_1/\partial \xi_3 = u, \partial w_2/\partial \xi_3 = v, \partial w_3/\partial \xi_3 = w.$$

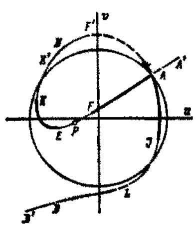

Fig. 1

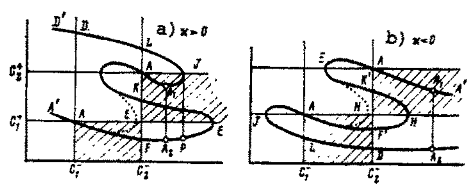

Fig.2

Three pairs of waves moving on both sides of the ξ_3 axis exist, which can be separated into quasilongitudinal and quasitransverse under moderate strains. Only the quasitransverse waves will be considered here. The equation of the shock adiabat of the quasitransverse shocks is obtained from the conservation law on a jump in /1,2/, i.e., the set of states of u, v, w in which it is possible to go by a jump from the initial state

$$(u^2 + v^2 - R^2)(Uv - Vu) - 2G(u - U)(v - V) = 0 \tag{1.1}$$
$$w = w^0 - 2b(u^2 + v^2 - R^2)$$
$$G = (\mu + {}^3/{}_2\gamma)(\varepsilon_{22} - \varepsilon_{11})/\varkappa, \quad R^2 = U^2 + V^2$$
$$\varkappa = \mu + (\mu + \beta + {}^3/{}_2\gamma)^2/(\lambda + \mu) - 2\xi$$

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$$2b = \lambda + 2\mu + \beta + {}^3/_2\gamma$$

This curve has the form shown in Fig. 1 in the uv plane. If all the quantities in (1.1) are referred to $\sqrt{G}$, then the parameter G drops out of the shock adiabat equation. This means that its dimensions are proportional to $\sqrt{G}$ while the shape and location relative to the axes depend on $U/\sqrt{G}$, $V/\sqrt{G}$.

The condition for no decrease in entropy and the conditions of evolutionarity /3/ have the form /1,2/

$$8\rho_0 T_0 (S - S_0) = -\varkappa [(u - U)^2 + (v - V)^2] (u^2 + v^2 - R^2) \geqslant 0 \tag{1.2}$$

$$\begin{aligned} &\text{a) } c_2^- \leqslant W \leqslant c_2^+, \ c_1^+ \leqslant W \\ &\text{b) } c_1^- \leqslant W \leqslant c_1^+, \ 0 \leqslant W \leqslant c_2^- \end{aligned} \tag{1.3}$$

Here c_i^+ and c_i^- are the characteristic velocities before and after the jump, respectively, where the numbering is selected so that $c_2 > c_1$. Analytic expressions for the characteristic velocities are presented in /1,2/.

For media with $\varkappa > 0$, condition (1.2) is satisfied within the circle $u^2 + v^2 = R^2$, on which the entropy is constant, $S = S_0 = \text{const}$. For media with $\varkappa < 0$ the condition is satisfied outside this circle. The evolutionarity conditions (1.3) turn out to be stronger for weak quasitransverse shocks in an elastic body, and isolate still narrower domains. They are displayed in the shock adiabat in Fig. 1 by solid lines for media with $\varkappa > 0$ and by dashed lines for media with $\varkappa < 0$.

Diagrams displaying the relationships between the shock velocity W and the velocity characteristics $c_{1,2}^-$ and $c_{1,2}^+$ which are plotted along the horizontal and vertical axes, rspectively /4/, are presented in Fig. 2 for $\varkappa > 0$ and $\varkappa < 0$. The shock adiabat is shown by a curve while the projection of each point on each of the axes is considered equal to the velocity of the discontinuity W corresponding to this point. The diagram is purely qualitative in nature and is for a graphical comparison between the velocity W and the characteristic velocities $c_{1,2}^\pm$. However, the velocities W, c_1^- and c_2^- can be plotted in real scales along the horizontal, as will indeed be assumed later. The initial point A in Fig. 1 is a point of selfintersection, i.e., there are two weak velocity jumps there. Consequently, the initial state A is represented by two points in the diagrams in Fig. 2. Corresponding points in Figs. 1 and 2 are denoted by identical letters.

The discontinuities satisfying the conditions (1.3)a, fast shocks, correspond to points of the shock adiabat that fall in the upper cross-hatched rectangle, while the discontinuities satisfying conditions (1.3)b, slow shocks, correspond to the shock adiabat points in the lower shaded rectangle. If the discontinuity is at a point on the evolutionary segment adjoining the point A, then we call such a jump a discontinuity of the first kind, otherwise, it is a discontinuity of the second kind. Depending on the values of $U/\sqrt{G}$ and $V/\sqrt{G}$ certaindiscontinuities of the second kind may be missing /2/. These cases are represented by dots in Fig. 2. For the sequel, we note that the velocity of the discontinuity has a maximum for $\varkappa > 0$ at the points E and J, and for $\varkappa < 0$ at the point H /2/.

Let us mention still another property of the states associated with the shock adiabat. It is possible to go from points in the sections DL, AH, AA' on the shock adiabat in media with $\varkappa > 0$ and in the sections AJ, AE for $\varkappa < 0$, to the state shown by the point $A\ (U, V)$ by a jump, just as from the initial points. The conservation laws and the condition of no decrease in entropy will evidently be satisfied here. Satisfaction of the evolutionary conditions is easily verified by using the diagram in Fig. 2.

2. Combination of two discontinuities. The case $\varkappa > 0$. We will show that the non-evolutionary part FE of the shock adiabat in the right lower rectangle (the conditions for its existence are given in /2/), corresponds to discontinuities that can be represented as the sequence of two evolutionary shocks, fast and slow moving at the same velocity one after the other. For all points of the arc FE a combination exists here that contains a fast wave of the second kind, and in addition a combination with a fast wave of the first kind exists for all points of the arc FE in which $W \leqslant \min\{W_E, W_J\}$.

Evidently all the conservation laws with the same mass, momentum, and energy flux values through unit area of the surface of discontinuity as on the first shock are satisfied on a discontinuity consisting of two successive shocks moving at the same velocity. Consequently, the state behind such a composite discontinuity lies on the shock adiabat referred to the initial state ahead of the first shock.

Let us consider two shocks, a fast shock propagating at a velocity W_1 and a slow shock moving at a velocity W_2 in the state behind the first wave. The quantities referring to the state behind the first shock will be provided with the subscript 1 and those referring to the state after the second wave will be given the subscript 2. The points A_1, A_2 correspond to these states in the uv plane, where if $W_1 = W_2$, then according to the above, the point A_2 in addition to the point A_1, also lies on the first original shock adiabat drawn through the point A as the initial one.

If the fast wave were a wave of the first kind and

$$W_1 \leqslant \min\{W_E, W_J\} \tag{2.1}$$

(here W_E, W_J are the velocities at the points J and E of the first shock adiabat), then a slow shock always exists that moves over the state behind the first wave at the same velocity $W_2 = W_1$.

To show this, we will first assume that the fast wave is of fairly low intensity.

According to (1.3)a and Fig. 2,a, its velocity will satisfy the strict inequalities $c_1^{(1)} < W < c_2^{(1)}$. The state behind this wave will differ slightly from the initial state. Then the velocity of the slow wave W_2 proceeding in the state A_1 behind the first shock can take any values between the characteristic velocities $c_1^{(1)}$ and $c_2^{(1)}$ ahead of this wave. This results from the fact that the shock adiabat changes slightly for a small change in the initial state and its point of intersection with the line $W = c_2^-$ does not vanish in Fig. 2,a. The point portraying the slow shock for which $W_2 = W_1$ will lie strictly within the rectangle in Fig. 2,a that corresponds to slow waves for a low first-wave intensity. Since W reaches a maximum at the point E, then for the discontinuity portrayed by a point with the rectange mentioned, on the same shock adiabat on which this point lies there are points corresponding to slow shocks with velocities smaller and greater than that selected.

Now, if W_1 changes (increases) continuously, then a slow shock can always be selected with a velocity W_2 equal to W until the point portraying the slow wave emerges on the boundary of the rectangle containing the domain of slow wave evolutionarity. If the equality $W = c_1^{(1)}$ is satisfied here (the right side of the rectangle), then the fast shock should correspond to the point J; if the equality $W = c_2^{(2)}$ (the upper boundary of the rectangle) were satisfied, then the result of the sequence of two jumps yields the point E, since only at this point of the initial shock adiabat does the equality $W = c_1^+$ hold. As long as the inequality (2.1) is satisfied, a slow shock will always exist that trails a fast shock and such that $W_2 = W_1$.

If the inequality (2.1) is satisfied for the whole arc FE, and this is possible for $W_J > W_E$, then the combination of evolutionary waves, as fast wave of the first kind and a slow wave, exists for each point of this segment. If $W_J < W_E$, then only points of the arc FP satisfy the inequality (2.1), where the point P is determined by the equality $W_P = W_J$. In this case points of the segment FP yield discontinuities representable in the form of a shock sequence: a fast one of the first kind and a slow one.

On the adiabat of a shock emerging from the point J, there is a point Q, unlike P, for which $W_Q = c_2^- = W_P$, where $W_P = W_J$. The point Q evidently belongs to the initial adiabat, and as is seen from Fig. 2,a, lies on the segment EK. When $W_J = W_E$, the points E, P, Q merge and the shock adiabats of the first and second waves touch one another at this point (Fig. 1). All the shock adiabats starting at points of the segment AJ intersect the initial adiabat twice, on the sections FE and EK.

When $W_J > W_E$, there is a point M on the segment AJ such that $W_M = W_E$. The shock adiabat emerging from the point M is tangent to the initial shock adiabat at the point E. All shock adiabats starting from points of the segment AM intersect the initial adiabat twice on different sides of the point E, while all shock adiabats starting from points of MJ do not intersect the initial shock adiabat.

We now consider the sequence of a fast wave of the second kind and a slow wave. We start with the fast wave corresponding to a point fairly close to the point E. Then the difference $W_1 - c_1^{(1)} > 0$ and can be taken as fairly small. For the state behind the first wave (as for any other state), a fairly weak slow shock can be indicated, whose velocity W_2 exceeds the characteristic velocity $c_1^{(1)}$ ahead of it by a given small quantity so that the equality $W_2 = W_1$ will be satisfied. The point portraying this weak slow shock will be within the rectangle of slow waves. Like the preceding case, a slow shock with W_2 equal to W_1 can always be chosen as W_1 changes (diminishes) as long as the point portraying this slow shock remains within the rectnagle containing the slow waves. It is easy to verify that emergence outside the boundary of this rectangle occurs if and only if the point representing the fast wave agrees with the point E. Therefore, a non-evolutionary discontinuity corresponding to any of the points of the arc FE can be represented in the form of a sequence of a fast shock of the second kind and a slow shock moving behind it at the same velocity.

3. **The case $\varkappa < 0$.** Like the preceding case, all points of the arc $F'H$ can be represented as a sequence of two evolutionary waves by two methods. One combination consists of a fast wave of the first kind and a slow wave of the first kind. The other combination consists of a fast wave of the second kind and a slow wave of the first kind. All the proof duplicate the discussion exactly for the case when $W_J > W_E$ in Sect.2 (Fig.2b).

All the points of the arc DD' correspond to discontinuities representable in the form of sequences of fast waves of the first kind and a slow wave of the second kind. Discussion resulting in such a conclusion can be performed by increasing the fast shock intensity. For a zero fast shock intensity, a slow wave arrives at the point D. Then the point portraying the slow shock departs within the rectangle and as the fast wave intensity increases further does not emerge outside the boundary of this rectangle, which proves the representability of the discontinuities corresponding to all points of the segment DD' in the form of the mentioned sequence of evolutionary shocks.

4. **Composite discontinuities.** Since the possibility of representing non-evolutionary discontinuities in the form of sequences of evolutionary discontinuities can be valuable not only for elasticity theory, we will discuss this question from a general viewpoint.

The conditions for the evolutionarity of a discontinuity that separates a domain of continuous solutions of a hyperbolic equations system include /3/ that the number of independent boundary conditions should equal the number of characteristics of different families that leave the discontinuity plus one. In solving the linearized problem this will permit the determination of the amplitude of the small perturbation waves leaving the discontinuity along the characteristics and the perturbation of the velocity of the discontinuity.

If there is a configuration of two evolutionary discontinuities moving with identical velocities, then the number of boundary conditions on both discontinuities should equal the number of all characteristics leaving both discontinuities plus two. If the velocities of the discontinuities do not agree with the characteristic velocities, then the number of characteristics leaving both discontinuities in the domain separating them will equal the order of the system, i.e., the number of independent variables characterizing the state between the discontinuities. If all quantities characterizing the state between the discontinuities are eliminated from the relationships on the discontinuities (at least mentally), then the number of remaining relationships connecting the quantities with the external sides of the system of discontinuities and the velocities W_1 and W_2 of the discontinuities will equal the number of characteristics departing to the outside plus two. Just as many relationships are evidently needed to find the perturbations outside the system of discontinuities and the perturbations of their velocities.

As already noted, if $W_1 = W_2$, then the sequence of such two discontinuities could be considered one discontinuity with all the conservation laws satisfied on it. This discontinuity is evidently non-evolutionary since, according to the above, the number of boundary conditions thereon exceeds the number of characteristics leaving it by two. Moreover, upon actual interaction with small perturbations, the velocities W_1 and W_2 can receive different increments, the jump is split and the perturbations cease to be small. If W_1 and W_2 are considered to be identically equal in the relations on the discontinuity (i.e., it is considered that the increments of these quantities are also equal), then the solution of the problem of interaction between the discontinuity and arbitrary small perturbations will not exist.

By reasoning similar to that presented above, it can be seen that if there are m evolutionary discontinuities moving at the same velocity $W_1 = W_2 = \ldots = W_m$, then the number of independent relationships on such a discontinuity from which quantities characterizing the state between the discontinuities are eliminated (or did not enter from the very beginning), should exceed by m the number of characteristics leaving such a combined discontinuity.

Hence, the following recommendation can be formulated. If there is a discontinuity on which the known relationships (following from the conservation laws, say) are too many for evolutionarity as a single discontinuity, then several evolutionary discontinuities should be sought which will turn into the discontinuity of interest to us when their velocities are equal.

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THE CONTACT-HYDRODYNAMIC PROBLEM OF LUBRICATION THEORY FOR ELASTIC BODIES WITH CRACKS *

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A mechanical model of lubricating solid bodies weakened by cracks is proposed. The model can be used to explain the reason for fatigue-induced crumbling of the surfaces. The presence of boundary and subsurface cracks is taken into account, and the interaction of the lubricant with elastic bodies within the cavities of boundary cracks is regarded as the most interesting aspect of the problem. Conditions are obtained characterizing the actual behaviour of the lubricant within the crack cavities, taking into account the pressure rise in the closed cavities completely filled with the lubricant and the possible onset of cavitation. The problem is reduced to a system of non-linear integro-differential and linear integral equations with additional conditions in the form of equations and inequalities.

The method of regular perturbations is used to study the state of weakly loaded elastohydrodynamic contact. In this case the problem is reduced to a sequence of purely hydrodynamic boundary value problems for the non-linear or linear ordinary differential equations, and elastic problems for the linear integral equations with one-sided constraints.

The effect of the temperature and lubricant on the contact stresses, taking the roughness of the bodies into account, was analysed in /1-3/ and the development of cracks and their influence on long-term fatigue in /4-7/.

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